INTRODUCTION GAME THEORY

Muddy Children, Unfaithful Husbands and the power of public announcement

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References and slides available at: http://staff.science.uva.nl/~peter/teaching/igt08.html

The Muddy Children puzzle

A group of children is playing outside and some of them have Collected dirt on their face. Each child can observe all other faces but not herself. Also the children are perfect reasoners.

The Father announces that there are some dirty children, and asks every child who knows herself to be dirty to step forwards.

For a while nothing happens, but at some point all dirty children step forwards.

How can this arrive ?

The induction proof

Induction on the number k of dirty children:

k=1 ; upon the announcement of the father the unique dirty child, observing nobody else with a dirty face, realizes that she must be the dirty one, and will step forwards.

k = n => k = n+1 ; a child X observing n dirty children considers two possibilities; either there exist n dirty children and by induction hypothesis they will eventually step forwards; if that doesn't happen since there are n+1 dirty children in total, X will realize that she is dirty herself, and so she will step forwards...

Sounds shakey, doesn't it ..... The snag is that there is no way this collective cycle of observation and inference is scheduled !
In the original puzzle the Father repeats his question until some children will react by stepping forwards, thus synchronizing the procedure.

Epistemic models

This puzzle is a standard intro into the area of updating Epistemic Models of the K5 logic of knowledge

Model : set of worlds, with a propositional valuation (propositoional model) +
for each agent an equivalence relation of indistiguishability

Agent X knows F at world w : $w \models K_X F$:
F is true in all worlds indistinguishable for that agent X from w

These epistemic operators can be nested

Everybody knows F : K_{X1} F and ... and K_{Xn} F

F is comon knowledge :
Infinitary conjuction of all nested $K_X K_Y..K_Z$ F

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Updating an epistemic model

Agent X updates his model upon learning an epistemic statement F by removing all worlds from the model where F is false.

Worlds which were indistinguishable and are preserved in the updated model remain indistinguishable.

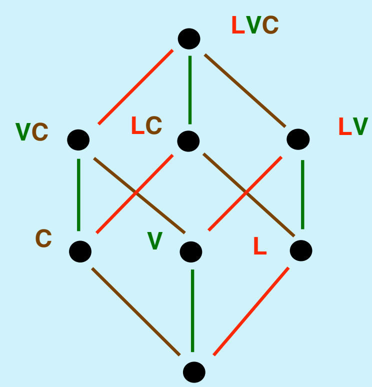
All formulas are evaluated again with respect to the new model.

Epistemic logic and models discussed in greater details in other Courses at ILLC.....

The model for three children

Lizzy
Vicky
Cathy

The label indicates at each world which children are dirty

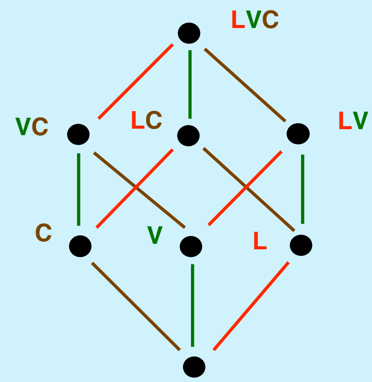


The model for three children

Lizzy
Vicky
Cathy

The initial statement by the father holds everywhere except at the bottom

Updating with the father's announcement means removing the bottom node.



The model for three children

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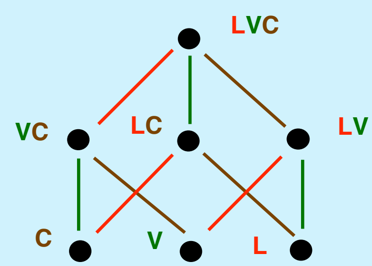
Updating with the father's announcement means removing the bottom node.

In the updated model at the three new bottom nodes some child knows herself to be dirty.



The model for three children

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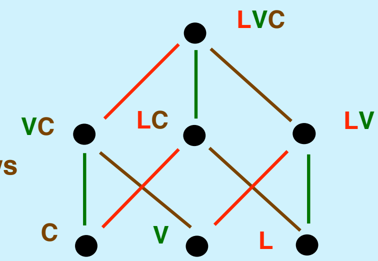

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Since no child steps forwards at the fathers reques the statement that some child knows herself to be dirty is false; these bottom worlds are removed

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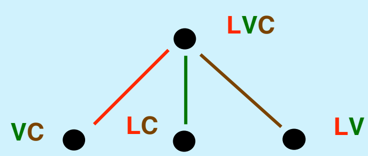

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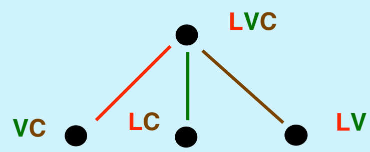

In the updated model at the three new bottom nodes both children know that they are dirty.

Since no child steps forwards at the repeated fathers request the statement that these children know that they are dirty is false; these bottom worlds are removed and only one world remains where they are all dirty

The model for three children

Lizzy
Vicky
Cathy

In the updated model at the three new bottom nodes both children know that they are dirty.

Since no child steps forwards at the repeated fathers request the statement that these children know that they are dirty is false; these bottom worlds are removed and only one world remains where they are all dirty

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The importance of scheduling

Binmore is silent on the topic of synchronizing the argueing of the agents.

The importance of this topic was illustrated already in 1985:

Yoram Moses, Danny Dolve & Joseph Y. Halpern:
Cheating Husbands and other stories; a case study of Knowledge, Action and Communication,

Preliminary version : PODC 4, 1985
Final version : Distributed Computing 1 (3), 1996, 167-176

The lost Kingdom of Atlantis

The lost Kingdom of Atlantis

The recently discovered scrolls by the great scholar Josephine describe the ongoing fight of the Atlantean queens against the recurring problem of male unfaithfullness.

Relevant facts:

Atlantis is a Matriarchate consisting of independent city states

Females require both a health and a intelligence test before obtaining a mariage permit (queens excepted)

Queens are truthfull ; Female citizens are Obedient

Gunshots are heared throughout the entire city

The proclamation

Proclamation of her highness Henrietta I , queen of Mamajorka, to be read in public at a meeting of all maried women at the main square of the city

There are (one or more) unfaithfull husbands in our city. Before this meeting you had no information concerning the status of your own husband, but you are perfectly informed about the faithfullness of all other husbands. Don't discuss this topic with anybody. However, if you discover at some point in time that your husband is unfaithfull, you must shoot him at midnight of the day you find out about it.

Henrietta I

Thirty nine silent nights went by, but on the fortieth night shots were heard.....

The induction proof

n : the number of unfaithfull husbands

Claim: If there are n unfaithfull husbands they will all be killed on the n-th night after the proclamation.

n=1 : The wife of the unique unfaithfull husband, seeing no other unfaithfull husbands knows that she must be the only one which is betrayed, and consequently kill him on the first night.

k = n => k = n+1 : With n+1 unfaithfull husbands, the betrayed wifes which observe only n unfaithfull ones, will see what happens on night n ; if their husband would be faithfull then the unfaithfull ones would be kiled that night by IH. So if nothing happens my husband must be guilty, and I must kill him at night n+1.

Connection with Common Knowledge

Before the proclamation every wife (though seeing n > 0) unfaithfull husbands can imagine that some other wife imagines that some other wife imagines that .... there are no unfaithfull husbands at all

Existence of unfaithfull husbands is not Common Knowledge before the proclamation; by the proclamation it becomes Common Knowledge, and all wifes adjust their model of the world accordingly

In the scenario that there are 40 unfaithfull husbands, every wife knows at the start that the first 38 nights will be silent; Still these quiet nights are crucial in the proof.

Binmore's treatment reads the proclamation as: after you have determined that your husband is unfaithful you must kill him.... But in this interpretation the proof is incorrect.

The curse of progress

Queen Henrietta I was highly regarded by her subjects for her wisdom and rightfulness. She ordered her daughters to persist in this battle against male infidelity.

She was eventually succeeded by her daughter Henrietta II . By that time public announcements at people gatherings were considered cumbersome and obsolete. Therefore Henrietta II introduced a mail system guaranteed to deliver any letter written by her to every household in the kingdom.

In her first letter she introduced and described in full details the working of this innovative system. Her second letter was an exact copy the original proclamation by Henrietta I .

Henrietta II suffered great disgrace and died in despair, ordering her daughters never to repeat her mistake….

Why Henrietta II was wrong

Claim: if there are at least 2 unfaithfull husbands they all will survive forever.

Proof: The start of the inductive proof goes on as before; upon arrival of the queens second letter the wife of an unique unfaithfull husband will discover that she is betrayed and will kill this husband the night following receiving the letter.

However, any wife observing a single unfaithfull husband can never determine whether that wife fails to shoot him because of the presence of yet another unfaithfull husband (I.E. mine…), or whether the letter of the queen has not yet reached that other women….

Thus the inductive proof breaks down.

The impact of asynchronic mail

A wife receiving the copy of the original proclamation will know that all other wives will receive it as well, and that the others also will eventually know that they all eventually will receive it, and…..

This is Eventual Common Knowledge, a infinitary conjuctions of modalities of the shape $S K_X S K_Y \ldots. F$, where S is a temporal modal operator (sometime in the future it will be the case that…)

The problem is to invent a corresponding update operator for the epistemic models (which now also require a temporal dimension…)

The possible fact that mail actually operates perfect (same day delivery) will not improve the situation !

Improving Service

Henrietta III succeeded her mother. She decided to upgrade the mail system, ensuring delivery before the evening of the next day following the day of sending. She realized that this woudn't have any impact as long as her subjects were unaware of this improvement.

Therefore she send first a letter explaining her subjects the upgrade of the mail system. Several days later she send a copy of the original proclamation of Henrietta I . The proclamation was undated since at that time the Calendar still had to be invented….

Henrietta III is considered to be more effective than her mother but is remembered for the great injustice she brought upon Mamajorka. She should have instructed her subjects to wait a few days before shooting….

Weakly synchronous mail

A Mail system is weakly asynchronous with bound b if mail is guaranteed to arrive no more than $b-1$ days after it is sent. So mail posted at day 1 arrives no later than day b . This system establishes b-eventual common knowledge defined as previously where S is replace by S_b denoting "within b days"

A new epistemic operator E : $E\ F$ meaning everybody knows F which amounts to the conjunction of $K_X\ F$ over all agents X

Claim: if there are n unfaithfull husbands and $E^n(W)$ becomes true for the original proclamation W then some cheated wife will kill her husband no later than n days after $E^n(W)$ becomes true.

Observation: with weakly asynchronous mail with bound b one has after b.n days of the original sending that $E^n(W)$ holds true.

Why E^n W suffices

proof of claim by induction:

$k=1$: the wife of the unique unfaithful husband will discover this fact as soon as she knows the content of the proclamation W, and kill him that very night.

$k = n \Rightarrow k = n+1$: a wife observing n unfaithfull husbands infers from $E^{n+1}(W)$ that these wives know $E^n(W)$. By IH this suffices for someone to be killed within n days. If that hasn't happened it must be the case that there is one more unfaithful husband around. Goodbye Johny…

When does it happen ?

A weakly synchronous mail system with bound b ensures that for every agentt who receives the initial message after an interval of b.n days it holds that E^n(“ the queen has mailed a copy of the proclamation”) ; however, according to the previous proof the agent should wait a number of days before shooting and observe the behavior of others who may have received the queens letter on a different day....

Claim: in the weakly synchronous mail system with bound b any wife knowing k unfaithful husbands will shoot her husband if k.b silent nights pass after reception of the queens letter.

The inductive proof

k = 1 : Upon reception of the letter a wife that observes no other unfaitfull husbands, knows everything she needs....

k = n => k = n+1 : if Cathy knows of n unfaithfull husbands, she observes the behavior of the corresponding wifes like E.G. Lizzy. Lizzy may have received the queens letter b-1 days later than Cathy, so if nothing has happened after b-1 + (n-1)b silent nights have passed it becomes evident that Lizzy must have observed more than n-1 unfaithfull husbands, indicating that Cathys husband himself is unfaithfull. No further evidence is needed. Note that the number of silent nights required becomes b-1 + b(n-1) + 1 = b.n

The unfairness

The proof entails that some unfaithfull husband will die, but contrary to the original problem other unfaithfull husbands may survive....

Cause: assume Cathy knows that Lizzy is betrayed, but knows no other unfaithful husbands. Cathy receives the letter on day x
When Cathy observes that Lizzy has shot her husband on night x + 1 there exist two possible scenario's:

Lizzy received the letter on day x and observing no other unfaithfull husbands shot her husband on night x + 1

Lizzy received the letter on day x – (b-1), and knowing that Cathy's husband is unfaithfull waited for b silent nights to follow, after which she fired.

So Cathy can never determine whether her husband is guilty or not.

Detailed Analysis

First significant day: the first day the queen's letter arrives at some betrayed wife. Assume that there are k+1 unfaithfull husbands.

The previous proof shows that after b.k silent nights following the first significant day, betrayed wifes having indeed received the letter on this first significant day will kill their husbands.

How about other betrayed wifes ? If they receive the letter on day x the first significant day may have occurred on any day y in the interval [x-(b-1),x+(b-1)] (and [x-(b-1), x] if they are betrayed).

Consider betrayed wife Cathy, who observes k unfaithfull ones.
If her husband is loyal, someone will get killed during the interval
[x-(b-1) + (k-1)b + 1,x+(b-1) + (k-1)b + 1] = [x + (k-2).b + 2, x + k.b].
Otherwise the first kill will occur during the interval
[x-(b-1) + kb + 1, x + kb + 1] = [x+(k-1)b + 2, x + kb + 1]
So shots observed during the interval [x+(k-1)b + 2, x + kb] are ambiguous and if Cathy receives the letter after the first significant day this will happen.

Delaying the kill

Josephine's cryptic last sentence has been analyzed to mean that the queen should have instructed her subjects to wait d days after discovering about the unfaithfullness of their husbands.

Claim : if the mail system is weakly synchronous with bound b and the directive is to delay the kill by d days, then with k+1 unfaithfull husbands the first shots will occur after k(b+d) silent nights following the first significant day.

Proof: similar to the previous case.

Delaying is useful

Assume that there are n unfaithfull ones.
F: first significant day , S: last silent night before the shots
The previous analysis shows $S = F + (n-1)(b+d) + d + 1$

Cathy receives on day x and observes k unfaithfull ones; two scenarios:

Cathy's husband is loyal: $x-b+1 \leq F \leq x+b-1$ so
$x-b+1+ (k-1)(b+d) + d + 1 \leq S \leq x+b-1+ (k-1)(b+d) + d + 1$

Cathy's husband is unfaithfull : $x-b+1 \leq F \leq x$ so
$x-b+1+ k(b+d) + d + 1 \leq S \leq x + k(b+d) + d + 1$

No ambiguity when
$x+b-1+ (k-1)(b+d) + d + 1 < x-b+1+ k(b+d) + d + 1$ which means
$b-1 < -b + 1 + b + d$ or $b-2 < d$. So the delay $d \geq b-1$ suffices.

Bribing the mailman

A wife may bribe the mailman to reveal the date the queen send her message. Since this happens in secret and the mailman is male no other women will know that this happens.

Claim: bribing the mailman eliminates possible ambiguity of hearing a shot.

Proof: the first essential day will occur before the b-1-st day after the queen mails her proclamation, and consequently, in the case of n unfaithfulls the firs shots will occur between day b(n-1)+1 and day bn. These intervals are disjoint, once there is no no uncertainty about what is day 0 .

Injustice breeds distrust

The next queen Henrietta IV decided to introduce a calendar to remove the ambiguity. Day 0 of the calendar was announced at a public gathering, supposed to be the last one ever arranged. In the future all mail would be dated and be delivered within b days.

A few weeks later Henrietta IV send a dated copy of the original proclamation to her subjects. When nothing happened for a period far beyond what here wise women had predicted, Henrietta IV concluded that because of the injustice of her mother some wives had lost their trust in the monarch and had become disobedient, and since that was comonly known amongst the wives the behavior of others no longer could be relied on.

Therefore Henrietta IV send out a follow up letter containing just one sentence: "At least one obedient women is married to an unfaithfull husband" . Henrietta IV restored her subjects' faith in monarchy.

The impact of distrust

Cathy observes a single unfaithfull husband married to Vicky.

Nothing has happened at a time Cathy knows that Vicky must have received the proclamation. Two scenario's are possible:

Vicky is the only betrayed wife, she knows this to be the case, but being disobedient she fails to perform her duty

Vicky is not the only betrayed wife, since my husband is unfaithfull as well

Note that if Cathy disregards the first possibility since she still assumes that all wives are obedient, Cathy will kill her loyal husband while the guily one walks....

The impact of the second letter

In this mail system, if it is common knowledge that at least one obedient wife is betrayed, all obedient betrayed wives will kill their hunsbands.

Proof: (induction on the number of unfaithfull ones n)

n=1 : the unique cheated wife must be obedient, so she will shoot upon receiving the second letter.

n=2 : Cathy is obedient and betrayed. She observes that Vicky is married to an unfaithfull one. Cathy now argues:

Vicky is the unique betrayed one, but then she will be obedient so before day b+1 she will shoot hence
If nothing has happened before day b+1 then Vicky is not unique and I must go for the kill (and it doesn't matter what Vicky will do)

Induction step

The induction hypothesis now reads:

If there are n unfaithfull husbands then the obedient betrayed wives will shoot their husbands at night $b + n - 1$.

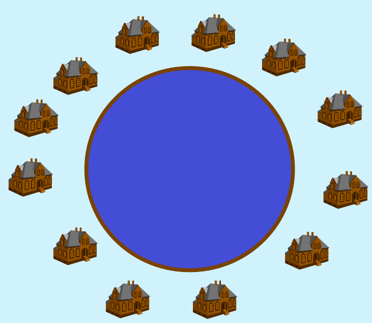
The cases $n=1$ and $n=2$ have been dealt with. The inductive argument reads: Obedient Cathy observing n unfaithfull ones knows that if her husband is loyal shots will be heared on the night $b + n - 1$. If that doesn't happen, she must kill therefore her husband on the following night $b + n = b + (n+1) - 1$

This will hold even if Cathy is the unique obedient wife.

Note that this system works much faster than the weakly synchronous system, even when all wives bribe the mailman.

Meanwhile in Mamaringa

In the neighbouring city state Mamaringa, the houses are build around a big lake, with a circular road connecting them.
The queens have always used a state of the art mail system where the mailman visits the houses in clockwise order starting at the Northern side of the lake.
All queens starting with Gracia II to Gracia IV have at some point of their reign issued a copy of the original proclamation. None of the queens fell in disgrace and none attained great honor. They are remebered as beeing cruel and unjust queens.



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Knowing the order of delivery

Claims:

a) Using asynchronous mail the last betrayed wife receiving the message will kill her husband; no other betrayed wife will do so.
b) Using weakly synchronous mail some betrayed wifes will kill but other wives may fail to do so
c) Using strongly synchronous mail some betrayed wifes will kill but other wives may fail to do so

The inductive proof for case a)

More specific claim: the last betrayed wife along the ring will kill her husband on night k after she receives the letter (k is the number of unfaithfull husbands she knows)

Proof: the case $k = 0$ is clear: the unique betrayed wife sees no unfaithfull ones at all and knows upon reception what to do.
Otherwise, a wife seeing only unfaithful ones in homes located before her own home, will hesitate between k or $k+1$ unfaithfull ones. In the first scenario the last preceding betrayed wife will shoot after $k-1$ nights after she has received the letter (which was on some earlier day). So after $k-1$ silent nights the situation is clear.
Wives observing unfaithful husbands farther down the road will observe that the last one will shoot some night. However, since it is unknown when the letter arrived, this will not yield information on the total number of unfaithful ones.

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Cases b) and c)

The earlier proofs on the synchronous systems remain valid, entailing that some unfaithfull husbands will be shot.
Injustice follows from the following scenario:

Cathy knows a single betrayed wife Lizzy, living farther down the road. The delivery delay $b = 2$. Cathy receives the letter on Sunday and hears Lizzy kill her husband on Monday night. The letter was posted on Sunday.

Possibilities:

Lizzy received on Sunday, waited for one night to see what Cathy would do with her unfaithful one and not hearing anything decided to shoot on Monday night. => Cathy's husband is guilty
Lizzy received on Monday and immediately knew what to do. => Cathy's husband is not guilty.

The impact of knowing the order of delivery

Is this additional information usefull or not ?

In the asynchronous case the last betrayed wife obtains usefull information, but injustice is introduced.

In the strongly synchronous case a system which is correct in Mamajorka now allows injustice in Mamaringa; the additional Information is dangerous.

The final improvement

Queen Henrietta V introduced the Express mail system: mail is guaranteed to be delivered the day it is posted. Evidently this by itself suffices to solve the recurring unfaithfullness problem but she wanted more: fast elimination.
So she first send out a letter introducing the new Express mail system. A few days later she posted a copy of the original proclamation with some additional instructions which allowed her subjects to shot in the air at midnight.

The syetem was very succesfull: the unfaithfull ones were eliminated from Mamajorka in just a few days.

The fast scheme

(a) A wife knowing of k unfaithful ones with $k \equiv 0 \pmod 3$ will shoot during the first night. If $k = 0$ her husband and otherwise in the air.

(b) If the first night was silent wives knowing k unfaithful ones with $k \equiv 1 \pmod 3$ will kill their husbands on the second night.
If the first night was noisy wives knowing k unfaithful ones with $k \equiv 2 \pmod 3$ will kill their husbands on the second night.

(c) If both the first and second nights were silent all wives kill their husbands on the third night.
If the first night was noisy and the second night was silent the shooters frommthe first night shoot again, but now to kill.

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Analysis

Observed numbers of unfaithfull ones will be adjacent, so mod 3 only two residue classes will be populated. Only if all husbands are unfaithful a single number k will be observed. Cases:

{0,1} mod 3 : wives observing 0 will shoot on the first night which suffices for the wives observing 1 to determine that their husbands are faithfull. The case of the observed value equal 0 is standard.

{0,2} mod 3 : wives observing 0 will shoot on the first night which suffices for the wives observing 2 to determine that their husbands are unfaithfull.

{1,2} mod 3 : First night is silent which suffices for the wives observing 1 (2) mod 3 to determine that their husband is guilty (not guilty).

The rule (c) covers the special case that all husbands are unfaithful.

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Optimality

There exists no similar protocol solving the problem in less than three nights.

Read the protocol as a list of instructions P(k) on how to behave if you initially know of k unfaithful ones.
If you are executing P(k) then someone else may execute P(k-1) or P(k+1) but not both.

P(k) may require to shoot on the first night, but in the air unless k = 0. Having shot in the air you should expect more shots that night, so you don't learn that night
If both P(k-1) and P(k+1) prescribe no shooting on the first night no information is collected by P(k).

So both with shots in the air and a silent first night some wife will require at least three nights.

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